Section 10.2- 10.3 Conic Sections: Hyperbolas and Ellipses

Ellipses and Hyperbolas have graphing equations that are very similar:

$$\frac{(x-h)^2}{a^2} \pm \frac{(y-k)^2}{b^2} = 1$$

The "center" is at (h , k) as you expect. The letters a and b control "how curvey" the figure is.

If we use the plus sign in the above equation we have an ellipse. If we use the minus sign in the equation above, we have a hyperbola.

Before I show how similar these are, we should consider what happens when we use a plus AND a and b are the same. We would have : $(x-h)^2 + (y-k)^2 = a^2$. That is the equation of a circle! A circle is really a special case of an ellipse!

I will show an example of an ellipse but in the end, I will change it to a hyperbola so we can see how different the graphs are but (again) how similar the equations are.

$$9x^2 + 4y^2 - 18x + 40y + 120 = 47$$

$$9x^2 - 18x + 4y^2 + 40y = 47 - 120$$

$$9\left(x^2 - 2x + \quad\right) + 4\left(y^2 + 10y + \quad\right) = -73$$

$$9\left(x^2 - 2x + \ 1 \ \right) + 4\left(y^2 + 10y + \ 25\right) = -73 + 9 + 100$$

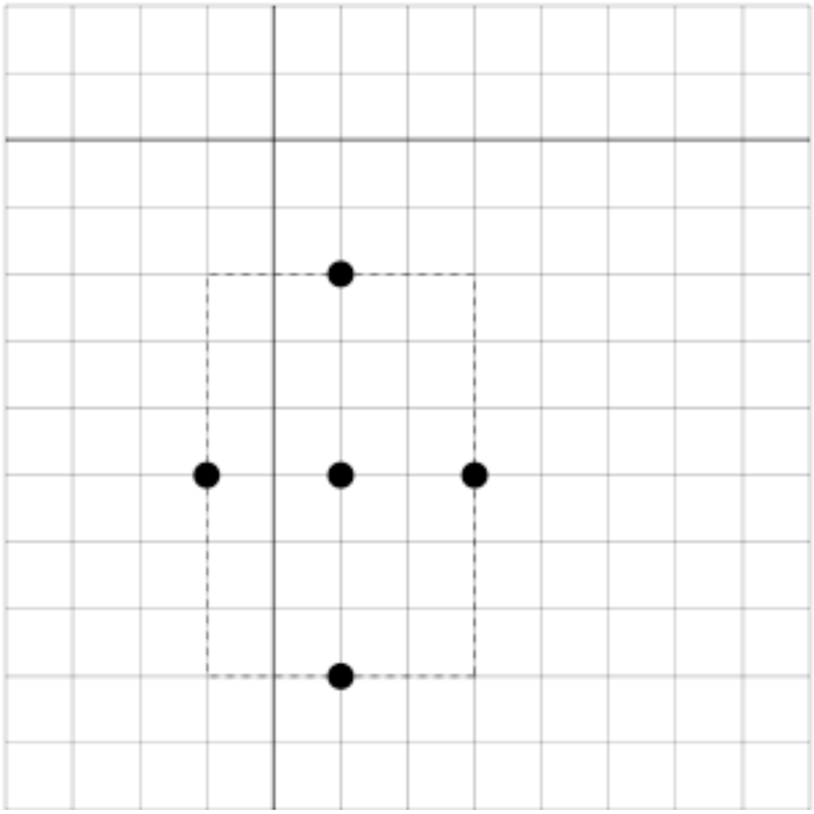
$$9(x-1)^2 + 4(y+5)^2 = 36$$

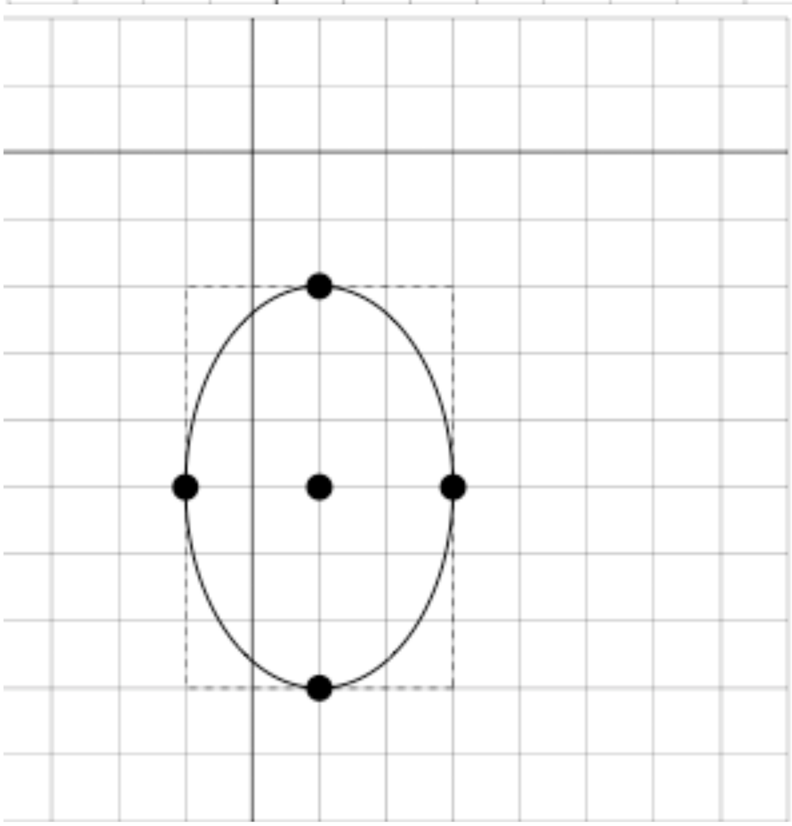
$$\frac{(x-1)^2}{2^2} + \frac{(y+5)^2}{3^2} = 1 \qquad\qquad \frac{(x-1)^2}{2^2} - \frac{(y+5)^2}{3^2} = 1$$

The left hand equation is the ellipse I started with. The right hand equation is the equation of a hyperbola. I will show the graph of the ellipse first.

The steps to graph:

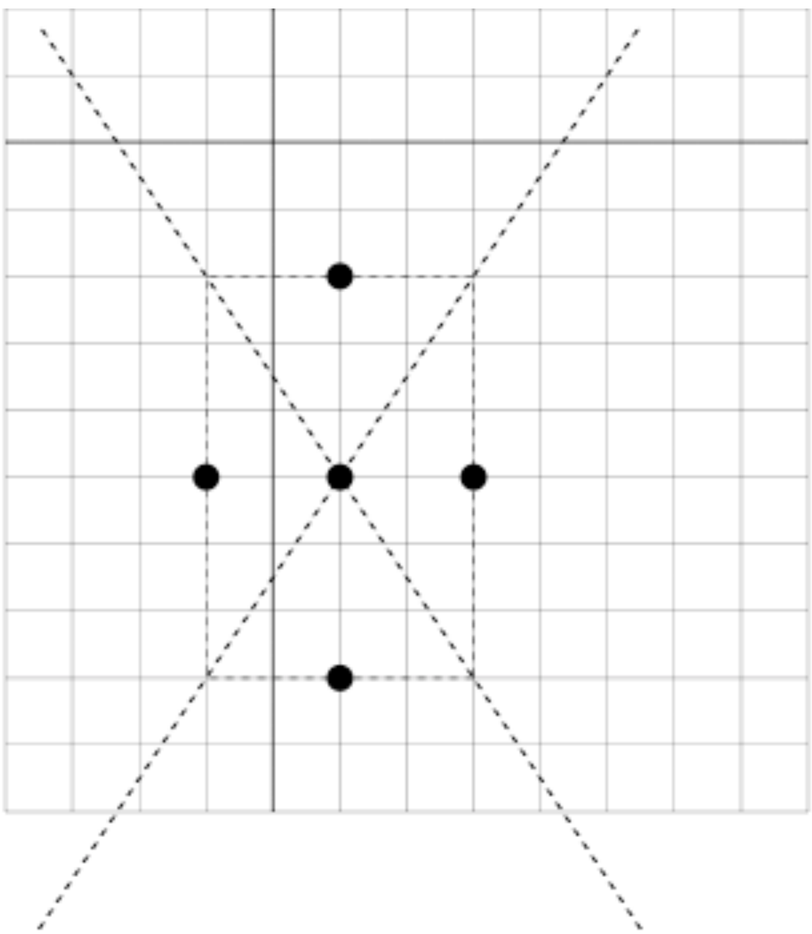
1. locate the "center" at (1, 5)
2. from the center count left (right) the "a" number (2). Make two dots.
3. from the center count up(down) the "b" number (3). Make two dots.
4. Draw a rectangle with those 4 dots on the sides.
5. the ellipse touches the box at those point. Sketch.



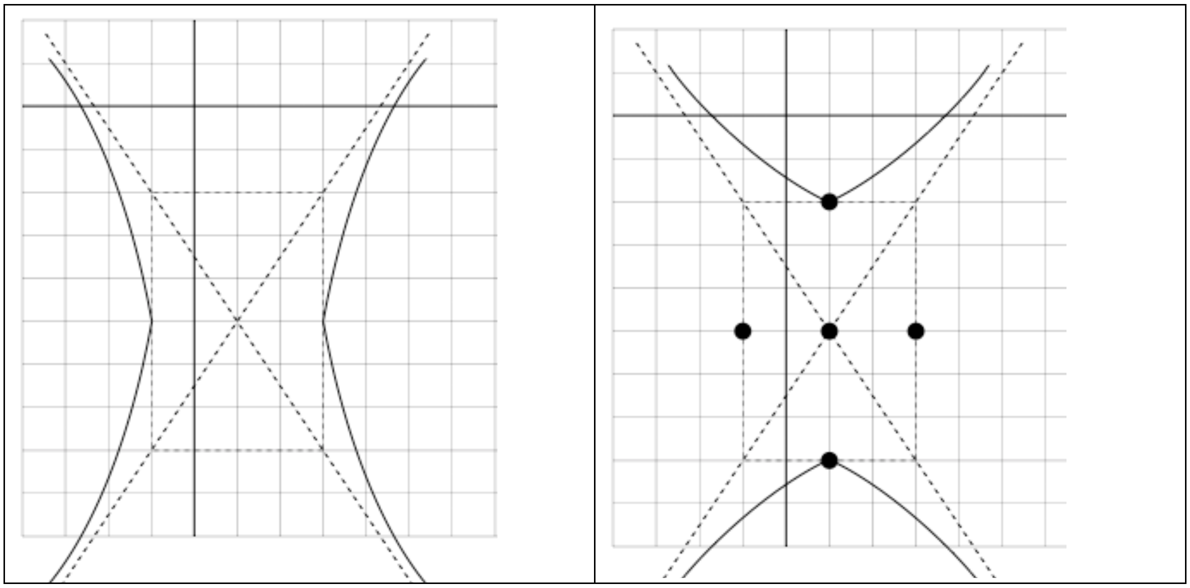


For the case of the hyperbola: $\frac{(x-1)^2}{2^2} - \frac{(y+5)^2}{3^2} = 1$

We draw the box exactly as we did for the ellipse.
Now we draw diagonal lines through the corners of the rectangular box.



We have 2 possibilities:
The hyperbola could look like either of these:



How do we know which graph is correct?

Remember the equation for this hyperbola: $\frac{(x-1)^2}{2^2} - \frac{(y+5)^2}{3^2} = 1$

IF $x = 1$ then the first term in the equation would be zero. The second term MUST be negative because it is composed of a negative, and a fraction that has both its top and bottom positive. Thus the left side of the equation is negative. The right side of the equation is positive so our equation ceases to be an equality. The conclusion is that x cannot equal 1.

On both of graphs above, draw the line $x = 1$. Which curve does *not* intersect with this line? The graph is on the left. Therefore, that is the proper picture for our hyperbola.

I suppose a more accurate "graphing form" should be: $\mp\frac{(x-h)^2}{a^2} \pm \frac{(y-k)^2}{b^2} = 1$.

If the term containing the y is the one positive, then the curve opens up and down.

I think it is informative to find the axis intercepts on this hyperbola.

Which axis will be crossed? The *x*-axis. When does a graph cross the *x*-axis? When the *y*-value is zero.

$$\frac{(x-1)^2}{2^2}-\frac{(y+5)^2}{3^2}=1$$
$$\frac{(x-1)^2}{2^2}-\frac{(0+5)^2}{3^2}=1$$
$$\frac{(x-1)^2}{4}-\frac{25}{9}=1$$
$$\frac{(x-1)^2}{4}=1+\frac{25}{9}$$
$$\frac{(x-1)^2}{4}=\frac{9}{9}+\frac{25}{9}$$
$$\frac{(x-1)^2}{4}=\frac{34}{9}$$
$$(x-1)^2=\frac{136}{9}$$
$$x-1=\pm\frac{\sqrt{136}}{\sqrt{9}} \qquad \textit{note}:\ 136=4\cdot 34 \text{ and } \sqrt{121}<136<\sqrt{144}$$
$$x-1=\pm\frac{2\sqrt{34}}{3} \qquad \textit{so} \qquad 11<\sqrt{136}<12$$
$$x=1\pm\frac{2\sqrt{34}}{3}$$
$$x=\frac{3\pm 2\sqrt{34}}{3}$$

We have *x*-intercepts at $\frac{3+11.\cdots}{3}=\frac{14.\cdots}{3}\approx$ just less than 5 and $\frac{3-11.\cdots}{3}=\frac{-8.\cdots}{3}\approx$ just greater than -3.

3 more examples:

A) $16x^2 + 9y^2 + 32x - 54y = 47$

1. Group x values and y values: $16x^2 + 32x + 9y^2 - 54y = 47$
2. For "completing the square" to work properly, the coefficients of the squared terms must be one. We will factor the 16 from the x terms and the 9 from the y terms.
   $16\left(x^2 + 2x + \quad\right) + 9\left(y^2 - 6y + \quad\right) = 47$
3. Completing the square adds a one to the x terms and a 9 to the y terms. The addition of those values adds 16 (16 times 1) and 81 (9 times 9) totaling 144.
   $16\left(x^2 + 2x + 1\right) + 9\left(y^2 - 6y + 9\right) = 47 + 16 + 81$
4. Collapsing the two terms gives us:
   $16(x+1)^2 + 9(y-3)^2 = 144$
5. The "graphing form for an ellipse" requires the equation to equal one.
   $$\frac{(x+1)^2}{9} + \frac{(y-3)^2}{16} = 1$$
   $$\frac{(x+1)^2}{3^2} + \frac{(y-3)^2}{4^2} = 1$$
6. The center of the ellipse is $(-1, 3)$. From that point we count left and right 3 because the 3 is "associated" with the x and we count up and down 4 because the 4 is "associated" with the y. Make the rectangular box marking the center of each of the "sides". The ellipse connects those 4 dots. In this example, the ellipse has its major (longer) axis vertically and the minor axis is horizontal.
7. To determine the axis intercepts requires that we know the curve crosses the x-axis when $y = 0$ and crosses the y-axis when $x = 0$.
   $16x^2 + 9y^2 + 32x - 54y = 47$

$x = 0$

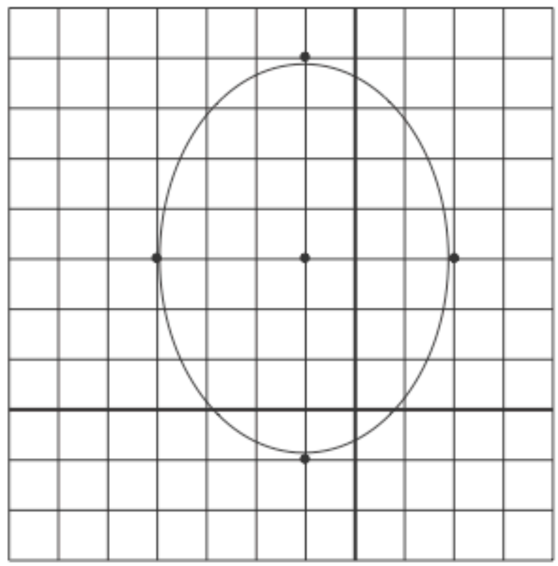
$$9y^2 - 54y = 47$$
$$9\left(y^2 - 6y + 9\right) = 47 + 81$$
$$9(y-3)^2 = 128$$
$$y - 3 = \pm\frac{\sqrt{128}}{3} = \pm\frac{8\sqrt{2}}{3}$$
$$y = 3 \pm \frac{8\sqrt{2}}{3}$$
$$\sqrt{121} < \sqrt{128} < \sqrt{144}$$
$$11.\cdots$$
$$\frac{11.\cdots}{3} \approx 3.\cdots$$

$y = 0$

$$16x^2 + 32x = 47$$
$$16\left(x^2 + 2x + 1\right) = 47 + 16$$
$$16(x+1)^2 = 63$$
$$x + 1 = \pm\frac{\sqrt{63}}{4} = \pm\frac{3\sqrt{7}}{4}$$
$$x = -1 \pm \frac{3\sqrt{7}}{4}$$
$$\sqrt{49} < \sqrt{63} < \sqrt{64}$$
$$7.\cdots$$
$$\frac{7.\cdots}{4} \approx 1.\cdots$$

8. So, y-intercepts values are $3 - 3.\cdots = -0.\cdots$
$3 + 3.\cdots = 6.\cdots$

9. So, x-intercepts values are $-1 - 1.\cdots = -2.\cdots$
$-1 + 1.\cdots = 0.\cdots$



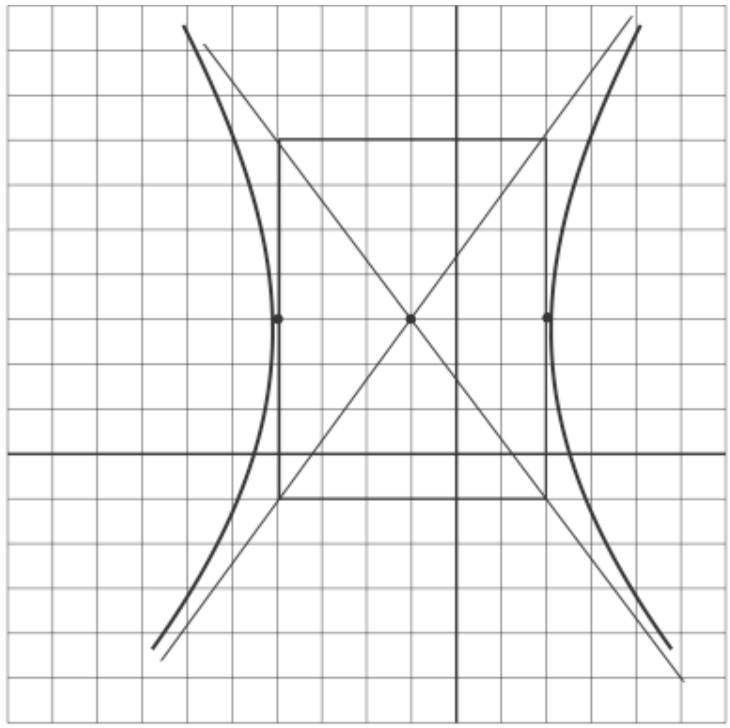
Example B

$16x^2 - 9y^2 + 32x + 54y = 209$

1. Group x values and y values: $16x^2 + 32x - 9y^2 + 54y = 209$
2. For "completing the square" to work properly, the coefficients of the squared terms must be one. We will factor the 16 from the x terms and the 9 from the y terms.
   $16\left(x^2 + 2x + \quad\right) - 9\left(y^2 - 6y + \quad\right) = 209$
3. Completing the square adds a one to the x terms and a 9 to the y terms. The addition of those values adds 16 (16 times 1) and 81 (9 times 9) totaling 144.
   $16\left(x^2 + 2x + \;1\;\right) - 9\left(y^2 - 6y + \;9\;\right) = 209 + 16 - 81$
4. Collapsing the two terms gives us:
   $16(x+1)^2 - 9(y-3)^2 = 144$
5. The graphing form requires the equation to equal one.
   $$\frac{(x+1)^2}{9} - \frac{(y-3)^2}{16} = 1$$
   $$\frac{(x+1)^2}{3^2} - \frac{(y-3)^2}{4^2} = 1$$
6. The center of the hyperbola is $(-1, 3)$. From that point we count left and right 3 because the 3 is "associated" with the x and we count up and down 4 because the 4 is "associated" with the y.
   Make the rectangular box marking the center of each of the "sides".
   Draw the asymptotes (diagonal lines through the corners of the box).7.
   $x \neq -1$ because if it did the first term would be zero, the second term negative and it would need to equal a positive one. Therefore, the hyperbola will not cross the value of x = – 1. To avoid this, the hyperbola would have to open left and right.

7. This hyperbola does not cross the y-axis. Other hyperbola could cross both of the axes.
   Finding the axis intersection is done in the same manner as in the previous example.

   Let $y = 0$ and solve for the x-axis intercepts.
   $16x^2 + 32x = 209$
   $16\left(x^2 + 2x + 1\right) = 209 + 16$
   $16(x+1)^2 = 225$
   $x = -1 \pm \frac{15}{4}$
   $x = -\frac{19}{4} \qquad x = \frac{11}{4}$



Example C

$9y^2 - 16x^2 - 54y - 32x = 79$

1. Group x values and y values: $9y^2 - 54y - 16x^2 - 32x = 79$
2. For "completing the square" to work properly, the coefficients of the squared terms must be one. We will factor the 16 from the x terms and the 9 from the y terms.

3. Completing the square adds a one to the x terms and a 9 to the y terms. The addition of those values adds 16 (16 times 1) and 81 (9 times 9) totaling 144.
   $9\left(y^2 - 6y + \ \ 9 \quad\right) - 16\left(x^2 + 2x + \ \ 1 \quad\right) = 79 + 81 - 16$
4. Collapsing the two terms gives us:
   $9(y-3)^2 - 16(x+1)^2 = 144$
5. The graphing form requires the equation to equal one.
   $\frac{(y-3)^2}{16} - \frac{(x+1)^2}{9} = 1$
   $\frac{(y-3)^2}{4^2} - \frac{(x+1)^2}{3^2} = 1$
6. The center of the hyperbola is $(-1, 3)$. From that point we count left and right 3 because the 3 is "associated" with the x and we count up and down 4 because the 4 is "associated" with the y.
   Make the rectangular box marking the center of each of the "sides".
   Draw the asymptotes (diagonal lines through the corners of the box).7.
   $y \neq 3$ because if it did the first term would be zero, the second term negative and it would need to equal a positive one. Therefore, the hyperbola will not cross the value of y =3. To avoid this, the hyperbola would have to open up and down.

7. This hyperbola does not cross the x-axis. Other hyperbola could cross both of the axes. Finding the axis intersection is done in the same manner as in the previous example.

$x = 0$

$$9y^2 - 54y = 79$$

$$9\left(y^2 - 6y + 9\right) = 79 + 81$$

$$9\left(y - 3\right)^2 = 160 \qquad \sqrt{144} < \sqrt{160} < \sqrt{169}$$

$$y - 3 = \pm\frac{4\sqrt{10}}{3} \qquad \frac{12.\cdots}{3}$$

$$y = 3 \pm \frac{4\sqrt{10}}{3} \qquad 4.\cdots$$

$$y = 3 - 4.\cdots = -1.\cdots \qquad y = 3 + 4.\cdots = 7.\cdots$$